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A VARIATIONAL FORMULATION FOR MODELING A COMPRESSIBLE AND TIME-DEPENDENT PROTIUM HYDROGEN FLUID FLOW

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Abstract

This article develops a variational formulation for modeling a compressible protium hydrogen fluid motion. The results are based on standard tools of calculus of variations and optimization theory. The context here addressed is essentially an Euler-Bernoullian one and includes also a new approximate Bernoulli-perfect-gas equation. Finally, it is worth highlighting, with the inclusion of such a new Bernoulli-perfect-gas equation, we have established a system of partial differential equations suitable for modeling a compressible case in fluid mechanics.

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1 Introduction

The first part of this article develops a variational formulation for modeling a gaseous hydrogen motion in an Euler-Bernoullian context.

Such results are based on tools of calculus of variations and related Lagrange multipliers approach. It is worth emphasizing, in the last section, we have obtained a system of partial differential equations including a new Bernoulli-perfect-gas equation originally presented in [1]. We believe such a new equation represents an important advance concerning the mathematical theory of fluid mechanics. It is also worth mentioning, such a system of partial differential equations we have obtained is similar to those previously known in the literature, however, with such a Bernoulli-perfect-gas equation replacing the standard ideal-gas equation.

Furthermore, we highlight the two main fields for such a system are a position one and a velocity one, which we specify below in the next lines.

As standard references in theoretical fluid mechanics we would cite [2, 3, 4, 5, 6, 7]. For related numerical methods we would mention [8, 9, 10, 11]. Concerning similar models, we would cite [12, 13, 14, 15, 16, 17].

Finally, details on the Sobolev spaces involved may be found in [18].

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and simply connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Generically, we consider a fluid motion initially modeled as a particle system one, through the field of position $\mathbf{r} : \Omega \times [0, t_f] \to \mathbb{R}^3$ and field of velocities $\mathbf{v}$ where $\mathbf{v}(\mathbf{r}(x,t),t)$ stands for the velocity at the position $\mathbf{r}(x,t)$ and time $t \in [0, t_f]$.

Moreover, the gravity field is denoted by

$$\mathbf{g} = -g\mathbf{k},$$

related to the $\mathbb{R}^3$ basis

$$\{\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)\}.$$

2 The mathematical model

In this section, we develop in details the concerning variational formulation for a compressible case.

We emphasize such a set $\Omega \subset \mathbb{R}^3$ stands for a control volume, in which along a time interval $[0, t_f]$, where $t \in [0, t_f]$ denotes time, a protium Hydrogen gas develops a compressible motion.

Here we recall a protium hydrogen atom comprises one electron and one proton (no neutrons). We also highlight the gaseous atomic protium Hydrogen is relatively rare on Earth however, may be found in great amount in stars such as the Sun.

Denoting by m_e the mass of one electron and by m_p the mass of one proton, we start by defining the following protium hydrogen atomic density

$$\hat{\rho}(x,y,z,t) = \left(|\phi_e(x,y,z,t)|^2 + |\phi_p(x,y,z,t)|^2\right)|\phi_A(y,z,t)|^2\frac{|\phi_M(z,t)|^2}{m_e+m_p} \quad (1)$$

The macroscopic fluid density is defined by

$$\rho = |\phi_M(z,t)|^2.$$

It is worth highlighting that generically, we denote

$$\begin{aligned} & L^2(\Omega\times[0,t_f]) \\ = & \left\{h:\Omega\times[0,t_f]\to\mathbb{R} \text{ measurable } : \int_\Omega\int_0^T |h(x,t)|^2\,dxdt<+\infty\right\}, \end{aligned} \quad (2)$$

$$\begin{aligned} L^2(\Omega\times[0,t_f];\mathbb{R}^3) = & \{h=(h_1,h_2,h_3):\Omega\times[0,t_f]\to\mathbb{R}^3 \\ & : h_k\in L^2(\Omega\times[0,t_f]),\ \forall k\in\{1,2,3\}\}, \end{aligned} \quad (3)$$

$$\begin{aligned} W^{1,2}(\Omega\times[0,t_f]) = & \{h:\Omega\times[0,t_f]\to\mathbb{R} \ : \ h\in L^2(\Omega\times[0,t_f]) \\ & h_t \text{ and } h_{x_j}\in L^2(\Omega\times[0,t_f]),\ \forall j\in\{1,2,3\}\}, \end{aligned} \quad (4)$$

where the partial derivatives h_{x_j} are understood in a distributional sense, $\forall j \in \{1,2,3\}$.

Moreover,

$$\begin{aligned} W^{1,2}(\Omega\times[0,t_f];\mathbb{R}^3) = & \{h=(h_1,h_2,h_3):\Omega\times[0,t_f]\to\mathbb{R}^3 \\ & : h_k\in W^{1,2}(\Omega\times[0,t_f]),\ \forall k\in\{1,2,3\}\} \end{aligned} \quad (5)$$

Similar notations are valid for other sets and other related complex and vectorial cases.

With such notations in mind, we define the concerning function spaces

$$\begin{aligned} V_1 = & \{\phi_e\in W^{1,2}(\Omega\times\Omega\times\Omega\times[0,t_f];\mathbb{C}) \ : \ \phi_e=0 \text{ on } \partial(\Omega\times\Omega\times\Omega)\times[0,t_f], \\ & \phi_e(x,y,z,0)=\hat{\phi}_0 \text{ and } \phi_e(x,y,z,t_f)=\hat{\phi}_f \text{ in } \Omega\times\Omega\times\Omega\} \end{aligned} \quad (6)$$

$$V_2 = W_0^{1,2}(\Omega\times\Omega\times[0,t_f];\mathbb{C}),$$

and

$$V_3 = \{\phi_M \in W^{1,2}(\Omega \times [0,t_f];\mathbb{C}) : \phi_M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0,t_f], \\ \phi_M(z,t) = \phi_{in}(z,t) \in \partial\Omega_{in} \times [0,t_f] \text{ and } \phi_M(z,0) = \phi_0 \text{ in } \Omega\}, \quad (7)$$

where

$$\partial\Omega = \partial\Omega_{in} \cup \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_{out}.$$

Moreover we suppose such a union is quasi-disjoint and through $\partial\Omega_{in}$ and $\partial\Omega_{out}$ are allowed the entering and exiting of gaseous hydrogen mass, respectively.

Remark 2.1. *In this text we denote*

$$\rho(z,t) = \int_\Omega \int_\Omega \hat{\rho}(x,y,z,t)\, dx\, dy$$

however, we consider ρ and $\hat{\rho}$ as two independent variables.

We define also

$$\hat{\rho}_e(x,z,y,t) = |\phi_e(x,y,z,t)|^2 |\phi_A(y,z,t)|^2 \frac{|\phi_M(z,t)|^2}{m_e + m_p}$$

$$\hat{\rho}_p(x,z,y,t) = |\phi_p(x,y,z,t)|^2 |\phi_A(y,z,t)|^2 \frac{|\phi_M(z,t)|^2}{m_e + m_p}$$

and the restrictions

$$\int_\Omega |\phi_e(x,y,z,t)|^2\, dx = m_e \text{ in } \Omega \times \Omega \times [0,t_f],$$

$$\int_\Omega |\phi_p(x,y,z,t)|^2\, dx = m_p \text{ in } \Omega \times \Omega \times [0,t_f]$$

and

$$\int_\Omega |\phi_A(y,z,t)|^2\, dy = 1, \text{ in } \Omega \times [0,t_f].$$

2.1 The position fields

We start by denoting the macroscopic hydrogen position field as

$$\mathbf{r} : \Omega \times [0,t_f] \to \mathbb{R}^3.$$

where

$$\mathbf{r}(z,t) = (X_1(z,t), X_2(z,t), X_3(z,t)), \text{ in } \Omega \times [0,t_f].$$

The atomic particle position field is defined by $\mathbf{r}_A : \Omega \times \Omega \times [0, t_f] \to \mathbb{R}^3$, where

$$\mathbf{r}_A(y, z, t) = \mathbf{r}(z, t) + \hat{\mathbf{r}}_A(y, \mathbf{r}(z, t)).$$

In a relativistic context, the electronic particle position field stands for

$$\mathbf{r}_e(x, y, z, t) = (ct, \hat{\mathbf{r}}_e(x, \mathbf{r}_A(y, z, t), \mathbf{r}(z, t))),$$

so that

$$\mathbf{r}_e(x, y, z, t) = (ct, \tilde{\mathbf{r}}_e(x, y, z, t)),$$

where

$$\tilde{\mathbf{r}}_e(x, y, z, t) = \hat{\mathbf{r}}_e(x, \mathbf{r}_A(y, z, t), \mathbf{r}(z, t)).$$

Similarly, the proton particle field stands for

$$\mathbf{r}_p(x, y, z, t) = (ct, \hat{\mathbf{r}}_p(x, \mathbf{r}_A(y, z, t), \mathbf{r}(z, t))),$$

so that

$$\mathbf{r}_p(x, y, z, t) = (ct, \tilde{\mathbf{r}}_p(x, y, z, t)),$$

where

$$\tilde{\mathbf{r}}_p(x, y, z, t) = \hat{\mathbf{r}}_p(x, \mathbf{r}_A(y, z, t), \mathbf{r}(z, t)).$$

Here $c > 0$ denotes the speed of light.

We define also the following function spaces

$$\begin{aligned} V_4 \;=\; & \{\mathbf{r} \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) \;:\; \mathbf{r} = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ & \text{and } \mathbf{r}(z, 0) = \mathbf{r}_0 \text{ in } \Omega\}, \end{aligned} \tag{8}$$

and

$$\tilde{\mathbf{r}}_e \in V_5 = W^{1,2}(\mathbb{R}^{10}; \mathbb{R}^3).$$

$$\tilde{\mathbf{r}}_p \in V_5$$

and

$$\hat{\mathbf{r}}_A \in V_6 = W^{1,2}(\mathbb{R}^7; \mathbb{R}^3).$$

2.2 About the system constraints

Denoting by $m_e(0)$ the total system mass at $t = 0$, we have the following constraint

$$\begin{aligned} m_e(0) &= \int_\Omega \rho(y,0)\, dy \\ &= \int_\Omega \rho(y,t)\, dy + \int_0^t \int_{\partial\Omega} \rho(y,\tau)\mathbf{v}\cdot\mathbf{n}\, dS\, d\tau \end{aligned} \tag{9}$$

where

$$\mathbf{v} = \mathbf{v}(\mathbf{r}(x,t),t).$$

Here $\mathbf{n}$ denotes outward normal field to $\partial\Omega = S$.

Relating the internal kinetics energy, in a relativistic context, we define the following functional

$$\begin{aligned} E^e_{C_1} &= -\frac{1}{2}\int_0^{t_f}\int_\Omega\int_\Omega\int_\Omega \frac{\hat\rho_e(x,y,z,t)}{\sqrt{1-\frac{v_e^2}{c^2}}}\frac{d\tilde{\mathbf{r}}_e}{dt}\cdot\frac{d\tilde{\mathbf{r}}_e}{dt}\, dx\, dy\, dz\, dt \\ &= \frac{1}{2}\int_0^{t_f}\int_\Omega\int_\Omega\int_\Omega \frac{\hat\rho_e}{\sqrt{1-\frac{v_e^2}{c^2}}}(c^2-v_e^2)\, dx\, dy\, dz\, dt \\ &= \frac{1}{2}\int_0^{t_f}\int_\Omega\int_\Omega\int_\Omega c\hat\rho_e\sqrt{c^2-v_e^2}\, dx\, dy\, dz\, dt \end{aligned} \tag{10}$$

where generically, we have denoted

$$a\cdot b = -a_0b_0 + \sum_{k=1}^{3} a_kb_k$$

for

$$a = (a_0,a_1,a_2,a_3)$$

and

$$b = (b_0,b_1,b_2,b_3) \in \mathbb{R}^4.$$

Similarly, we may define

$$\begin{aligned} E^p_{C_1} &= -\frac{1}{2}\int_0^{t_f}\int_\Omega\int_\Omega\int_\Omega \frac{\hat\rho_p(x,y,z,t)}{\sqrt{1-\frac{v_p^2}{c^2}}}\frac{d\tilde{\mathbf{r}}_p}{dt}\cdot\frac{d\tilde{\mathbf{r}}_p}{dt}\, dx\, dy\, dz\, dt \\ &= \frac{1}{2}\int_0^{t_f}\int_\Omega\int_\Omega\int_\Omega \frac{\hat\rho_p}{\sqrt{1-\frac{v_p^2}{c^2}}}(c^2-v_p^2)\, dx\, dy\, dz\, dt \\ &= \frac{1}{2}\int_0^{t_f}\int_\Omega\int_\Omega\int_\Omega c\hat\rho_p\sqrt{c^2-v_p^2}\, dx\, dy\, dz\, dt \end{aligned} \tag{11}$$

For the interacting and self-interacting energies, we define the following functionals,

$$
\begin{aligned}
& E^e_{in} \\
= \; & \frac{\alpha_e}{2} \int_0^{t_f} \int_\Omega \int_\Omega \int_\Omega \int_\Omega \int_\Omega \int_\Omega \frac{|\phi_e(x,y,z,t)|^2 |\phi_e(\tilde{x},\tilde{y},\tilde{z},t)|^2}{|\mathbf{r}_e(x,y,z,t) - \mathbf{r}_e(\tilde{x},\tilde{y},\tilde{z},t)|} \\
& \times |\phi_A(y,z,t)|^2 |\phi_A(\tilde{y},\tilde{z},t)|^2 \\
& \times |\phi_M(z,t)|^2 |\phi_M(\tilde{z},t)|^2 \, dx \, dy \, dz \, d\tilde{x} \, d\tilde{y} \, d\tilde{z} \, dt
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
& E^p_{in} \\
= \; & \frac{\alpha_p}{2} \int_0^{t_f} \int_\Omega \int_\Omega \int_\Omega \int_\Omega \int_\Omega \int_\Omega \frac{|\phi_p(x,y,z,t)|^2 |\phi_p(\tilde{x},\tilde{y},\tilde{z},t)|^2}{|\mathbf{r}_p(x,y,z,t) - \mathbf{r}_p(\tilde{x},\tilde{y},\tilde{z},t)|} \\
& \times |\phi_A(y,z,t)|^2 |\phi_A(\tilde{y},\tilde{z},t)|^2 \\
& \times |\phi_M(z,t)|^2 |\phi_M(\tilde{z},t)|^2 \, dx \, dy \, dz \, d\tilde{x} \, d\tilde{y} \, d\tilde{z} \, dt
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
& E^{ep}_{in} \\
= \; & \frac{\alpha_{ep}}{2} \int_0^{t_f} \int_\Omega \int_\Omega \int_\Omega \int_\Omega \int_\Omega \int_\Omega \frac{|\phi_e(x,y,z,t)|^2 |\phi_p(\tilde{x},\tilde{y},\tilde{z},t)|^2}{|\mathbf{r}_e(x,y,z,t) - \mathbf{r}_p(\tilde{x},\tilde{y},\tilde{z},t)|} \\
& \times |\phi_A(y,z,t)|^2 |\phi_A(\tilde{y},\tilde{z},t)|^2 \\
& \times |\phi_M(z,t)|^2 |\phi_M(\tilde{z},t)|^2 \, dx \, dy \, dz \, d\tilde{x} \, d\tilde{y} \, d\tilde{z} \, dt.
\end{aligned} \tag{14}
$$

On the other hand, we define

$$
\begin{aligned}
& E^e_M \\
= \; & \frac{\gamma_e}{2} \int_0^{t_f} \int_\Omega \int_\Omega \int_\Omega \nabla(\phi_e(x,y,t)\phi_A(y,z,t)\phi_M(z,t)) \\
& \cdot \nabla(\phi^*_e(x,y,z,t)\phi^*_A(y,z,t)\phi^*_M(z,t)) \, dx \, dy \, dz \, dt
\end{aligned} \tag{15}
$$

for an appropriate $\gamma_e > 0$ and,

$$
\begin{aligned}
& E^p_M \\
= \; & \frac{\gamma_p}{2} \int_0^{t_f} \int_\Omega \int_\Omega \int_\Omega \nabla(\phi_p(x,y,t)\phi_A(y,z,t)\phi_M(z,t)) \\
& \cdot \nabla(\phi^*_p(x,y,z,t)\phi^*_A(y,z,t)\phi^*_M(z,t)) \, dx \, dy \, dz \, dt
\end{aligned} \tag{16}
$$

for an appropriate $\gamma_p > 0$.

Concerning the restrictions, we define also the following functionals,

$$
\begin{aligned}
&J_{Aux_1} = \\
&\int_0^{t_f} \int_\Omega \int_\Omega E_1(y,z,t) \left(\int_\Omega (|\phi_e(x,y,z,t)|^2 \, dx - m_e \right) dy \, dz \, dt \\
&+ \int_0^{t_f} \int_\Omega \int_\Omega E_{1a}(y,z,t) \left(\int_\Omega (|\phi_p(x,y,z,t)|^2 \, dx - m_p \right) dy \, dz \, dt \\
&+ \int_0^{t_f} \int_\Omega E_{1b}(z,t) \left(\int_\Omega (|\phi_A(y,z,t)|^2 \, dy - 1 \right) dz \, dt, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
J_{Aux_2} \;=\; & \int_0^{t_f} E_2(t) \left(\int_\Omega \rho(y,0) \, dy \right. \\
& \left. - \int_\Omega \rho(y,t) \, dy - \int_0^t \int_{\partial\Omega} \rho(y,\tau) \mathbf{v} \cdot \mathbf{n} \, dS \, d\tau \right) dt \qquad (18)
\end{aligned}
$$

$$
J_{Aux_3} = \int_0^{t_f} \int_\Omega E_3(x,t) \left(\mathbf{v}(\mathbf{r}(x,t),t) - \frac{\partial \mathbf{r}(x,t)}{\partial t} \right) dx \, dt,
$$

$$
J_{Aux_4} \;=\; -\int_0^{t_f} \int_\Omega \hat{P} \left(\frac{d\rho}{dt} + \rho \, \mathrm{div} \, \mathbf{v} \right) dy \, dt, \qquad (19)
$$

Moreover, considering the pressure and temperature scalar fields, we define

$$
P = \frac{d(\hat{P}\rho)}{dt},
$$

where

$$
\hat{P} \in V_5
$$

and

$$
V_5 = \{ \hat{P} \in W^{1,2}(\Omega \times [0,t_f]) \; : \; P = P(\hat{P},\rho) = P_0, \text{ on } \partial\Omega \times [0,t_f] \}.
$$

$$
(C_v)\rho T \;=\; \hat{E}_C^e + \hat{E}_C^p \qquad (20)
$$

where

$$
\hat{E}_C^e = \frac{1}{2} c \int_\Omega \int_\Omega \hat{\rho}_e(x,y,z,t) \sqrt{c^2 - v_e^2} \, dxdy,
$$

and

$$
\hat{E}_C^p = \frac{1}{2} c \int_\Omega \int_\Omega \hat{\rho}_p(x,y,z,t) \sqrt{c^2 - v_p^2} \, dxdy,
$$

for an appropriate real constant $C_v > 0$.

Here we define the entropy differentials

$$dS_e = -K_B \frac{\hat{\rho}_e}{\hat{\rho}} \ln\left(\frac{\hat{\rho}_e}{e\,\hat{\rho}}\right)\, dxdydzdt,$$

and

$$dS_p = -K_B \frac{\hat{\rho}_p}{\hat{\rho}} \ln\left(\frac{\hat{\rho}_p}{e\,\hat{\rho}}\right)\, dxdydzdt.$$

Moreover, we define

$$S_e = -K_B \frac{\hat{\rho}_e}{\hat{\rho}} \ln\left(\frac{\hat{\rho}_e}{e\,\hat{\rho}}\right),$$

and

$$S_p = -K_B \frac{\hat{\rho}_p}{\hat{\rho}} \ln\left(\frac{\hat{\rho}_p}{e\,\hat{\rho}}\right)$$

and

$$E_S = \int_0^{t_f} \int_\Omega \int_\Omega \int_\Omega T(dS_e + dS_p),$$

for an appropriate constant $K_B > 0$.

With such results in mind, defining $S = \int_\Omega \int_\Omega (S_e + S_p)\, dx\, dy$, we define also the restriction

$$\rho(C_v)\frac{dT}{dt} = K\nabla^2 T$$
$$-P \text{ div } \mathbf{v} + \frac{d(TS)}{dt}, \tag{21}$$

for an appropriate real constant $K > 0$.

At this point, we define the following function space

$$\begin{aligned} V_6 \;=\; & \{T \in W^{1,2}(\Omega \times [0,t_f]) \;:\; T(z,t) = T_1(z,t) \\ & \text{on } \partial\Omega \times [0,t_f] \text{ and } T(z,0) = T_0(z) \text{ in } \Omega\}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} V_7 \;=\; & \{\mathbf{v} \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^3) \;:\; \mathbf{v} = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0,t_f] \\ & \text{and } \mathbf{v}(z,0) = \mathbf{v}_0 \text{ on } \Omega\}. \end{aligned} \tag{23}$$

With such restrictions and definitions in mind, we define the following functionals,

$$\begin{aligned} & J_{Aux_5} \\ = \; & \int_0^{t_f} \int_\Omega E_5(z,t)\left((C_v)\rho T - (\hat{E}_C^e + \hat{E}_C^p)\right)\, dz\, dt \end{aligned} \tag{24}$$

and

$$J_{Aux_6} = \int_0^{t_f} \int_\Omega E_6(z,t) \left(\rho(C_v) \frac{dT}{dt} - K\nabla^2 T + P \text{ div } \mathbf{v} - \frac{d(TS)}{dt} \right) dz\, dt. \quad (25)$$

Finally, we define the following functionals,

$$J_{Aux_7} = \int_0^{t_f} \int_\Omega E_{50}(z,t) \times \left(\rho(z,t) - \int_\Omega \int_\Omega (\hat{\rho}_e(x,y,z,t) + \hat{\rho}_p(x,y,z,t))\, dxdy \right) dz\, dt \quad (26)$$

and

$$J_{Aux_8} = \int_0^{t_f} \int_\Omega E_{51}(z,t)(\rho(z,t) - |\phi_M|^2)\, dzdt.$$

3 The main functional

Here we define the functional J by

$$J = \frac{1}{2} \int_0^{t_f} \int_\Omega \rho \mathbf{v} \cdot \mathbf{v}\, dy\, dt - \int_0^T \int_\Omega \rho \mathbf{g} \cdot \mathbf{r}\, dy\, dt. \quad (27)$$

The final variational formulation J_{total} for such an ionization process stands for

$$J_{total} = J - E^e_{C_1} - E^p_{C_1} + E^e_{in} + E^p_{in} + E^{ep}_{in} + E^e_M + E^p_M + \sum_{k=1}^{8} J_{Aux_k} + E_S. \quad (28)$$

Observe that the extremal equation

$$\frac{\partial J_{total}}{\partial \rho} = \mathbf{0}$$

stands for

$$\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + \frac{\partial J_{Aux_2}}{\partial \rho} + \frac{\partial J_{Aux_5}}{\partial \rho} + \frac{\partial J_{Aux_6}}{\partial \rho} + \frac{d\hat{P}}{dt} - \hat{P} \text{ div } \mathbf{v} - \mathbf{g} \cdot \mathbf{r} + E_{50}(z,t) + E_{51}(z,t) = \mathbf{0}. \quad (29)$$

Thus, we have

$$\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{d\hat{P}}{dt}-\hat{P}\ \mathrm{div}\ \mathbf{v}-\mathbf{g}\cdot\mathbf{r}+E_{51}(z,t)+\tilde{E}_1=\mathbf{0}. \tag{30}$$

for an appropriate function $\tilde{E}_1$.

Observe also that

$$P=\frac{d(\hat{P}\rho)}{dt},$$

so that

$$P=\frac{d\hat{P}}{dt}\rho+\hat{P}\frac{d\rho}{dt}.$$

Thus,

$$\frac{d\hat{P}}{dt}=\frac{P}{\rho}-\frac{\hat{P}}{\rho}\frac{d\rho}{dt}.$$

From such last results, we may infer that

$$\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{P}{\rho}-\frac{\hat{P}}{\rho}\frac{d\rho}{dt}-\hat{P}\ \mathrm{div}\ \mathbf{v}-\mathbf{g}\cdot\mathbf{r}+E_{51}(z,t)+\tilde{E}_1=\mathbf{0}, \tag{31}$$

so that,

$$\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{P}{\rho}-\mathbf{g}\cdot\mathbf{r}+E_{51}(z,t)+\tilde{E}_1=\mathbf{0}, \tag{32}$$

Moreover, from the extremal equation

$$\frac{\partial J_{total}}{\partial\hat{\rho}_e}=\mathbf{0},$$

we obtain

$$-c\sqrt{c^2-v_e^2}-E_{51}(z,t)+\tilde{E}_2=0,$$

and from the extremal equation

$$\frac{\partial J_{total}}{\partial\hat{\rho}_p}=\mathbf{0},$$

we obtain

$$-c\sqrt{c^2-v_p^2}-E_{51}(z,t)+\tilde{E}_3=0,$$

for appropriate functions $\tilde{E}_2$ and $\tilde{E}_3$.

From these last two equations, neglecting $\tilde{E}_2$ and $\tilde{E}_3$, we obtain

$$-\int_\Omega\int_\Omega \hat{\rho}_e c\sqrt{c^2-v_e^2}\,dxdy-\int_\Omega\int_\Omega \hat{\rho}_p c\sqrt{c^2-v_p^2}\,dxdy$$
$$-E_{51}(z,t)\left(\int_\Omega\int_\Omega \hat{\rho}_e\,dxdy+\int_\Omega\int_\Omega \hat{\rho}_p\,dxdy\right)\approx 0, \tag{33}$$

so that

$$-\int_\Omega\int_\Omega \hat{\rho}_e c\sqrt{c^2-v_e^2}\,dxdy-\int_\Omega\int_\Omega \hat{\rho}_p c\sqrt{c^2-v_p^2}\,dxdy-E_{51}(z,t)\rho\approx 0,$$

that is,

$$E_{51}(z,t)\rho\approx -C_v\rho T.$$

Moreover from (32), neglecting $\tilde{E}_1$, we have

$$\frac{\rho}{2}\mathbf{v}\cdot\mathbf{v}+P-\rho\mathbf{g}\cdot\mathbf{r}\approx -E_{51}(z,t)\rho=K_5\rho T \tag{34}$$

for an appropriate constant $K_5>0$.

From such a result we have also obtained the following Bernoulli-perfect-gas equation

$$\frac{P}{\rho}+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gX_3=K_5T+C_3,$$

for an appropriate $C_3\in\mathbb{R}$ and an appropriate $K_5>0$.

In summary, we have got the following Bernoulli-perfect-gas equation

$$P+\frac{\rho}{2}\mathbf{v}\cdot\mathbf{v}+\rho gX_3=K_5\rho T+C_3\rho,$$

for real constant C_3.

Finally, from the variational of J_{total} in $\mathbf{v}$, we obtain

$$\rho\mathbf{v}+E_3+\nabla(\hat{P}\rho)+C_5=\mathbf{0},$$

for an appropriate function C_5. The variation of J_{total} in $\mathbf{r}$, stands for

$$-\rho\mathbf{g}+\frac{dE_3}{dt}+C_7=\mathbf{0},$$

for an appropriate function C_7. Joining the pieces we have got

$$\rho\mathbf{g}+\frac{d(\rho\mathbf{v})}{dt}+\frac{d(\nabla(\hat{P}\rho))}{dt}-C_{10}=\mathbf{0},$$

so that

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \frac{\partial(\rho \mathbf{v})}{\partial X_j}\frac{\partial X_j}{\partial t} + \nabla P + \rho \mathbf{g} = C_{10},$$

where

$$C_{10} = C_7 - \frac{dC_5}{dt}.$$

This just an approximate version of the well known Euler system.

Moreover, from the variation of J_{total} in $\hat{P}$, we have,

$$\frac{d\rho}{dt} + \rho \text{ div } \mathbf{v} - C_8 = 0,$$

for an appropriate function C_8.

Hence, through the Eulerian identification

$$X_k(y,t) \approx x_k, \ \forall k \in \{1,2,3\},$$

neglecting C_5, C_7, C_8, we have obtained the following approximate macroscopic system,

1. Euler Equations:

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \frac{\partial(\rho \mathbf{v})}{\partial x_j} v_j + \nabla P = -\rho \mathbf{g}.$$

2. Continuity equation:

$$\frac{d\rho}{dt} + \rho \text{ div } \mathbf{v} = 0,$$

where we have generically (with a not relabeled notation) denoted

$$\rho = \rho(\mathbf{r}(x,t),t),$$

so that with the Eulerian identification

$$X_k(y,t) \approx x_k, \ \forall k \in \{1,2,3\},$$

we have

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_j} v_j.$$

3. A Temperature T equation:

$$\rho(C_v)\frac{dT}{dt} = K\nabla^2 T$$
$$-P \text{ div } \mathbf{v} + \frac{d(TS)}{dt}. \tag{35}$$

4. A Bernoulli-perfect-gas type equation:

$$P + \frac{\rho}{2}\mathbf{v}\cdot\mathbf{v} + \rho g z = K_5\rho T + C_3\rho.$$

Remark 3.1. *We highlight the functions C_5, C_7, C_8 and C_{10} may be exactly obtained from the exact previous variations of J_{total}.*

4 A numerical example

Let $\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2$ and let $[0, t_f]$ be a time interval.

Consider the following time-dependent, compressible approximate Navier Stokes system, which includes viscosity effects (see [10] for details), where

$$\begin{cases} (\rho u)_t = \nu\nabla^2 u - u(\rho u)_x - v(\rho u)_y - P_x, \\ (\rho v)_t = \nu\nabla^2 v - u(\rho v)_x - v(\rho v)_y - P_y, \\ \rho_t + (\rho u)_x + (\rho v)_y = \varepsilon\nabla^2\rho \\ \rho C_v T_t = K_5\nabla^2 T - P(u_x + v_y) + \Phi, \\ P = -\rho(u^2+v^2)/2 + K_3\rho T - K_3\rho, \qquad \text{in } \Omega\times[0,t_f]. \end{cases} \tag{36}$$

Here,

$$\Phi = 2C_4(u_x^2 + v_y^2) + C_4(u_y + v_x)^2 - (2/3)C_4(u_x + v_y)^2,$$

$\nu > 0$ is a small constant to be specified, $\mathbf{u} = (u, v, T, \rho) \in W^{1,2}(\Omega\times[0,t_f];\mathbb{R}^4)$ where u denotes the fluid velocity in the direction $\mathbf{i} = (1,0) \in \mathbb{R}^2$, v denotes the fluid velocity in the direction $\mathbf{j} = (0,1) \in \mathbb{R}^2$, T is the scalar temperature field, ρ is the fluid density field and P is a concerning pressure field, with the boundary conditions,

$$u(0,y) = \hat{u}_0(y), \text{ in } [0,1],$$

$$u(x,0) = 0, \text{ in } [0,1],$$
$$u(x,1) = 0, \text{ in } [0,1],$$
$$u_x(1,y) = 0, \text{ in } [0,1],$$
$$v(0,y) = 0, \text{ in } [0,1],$$
$$v(x,0) = 0, \text{ in } [0,1],$$
$$v(x,1) = 0, \text{ in } [0,1],$$
$$v_x(1,y) = 0, \text{ in } [0,1],$$
$$\rho \equiv 1, \text{ on } \partial\Omega,$$
$$T \equiv 1, \text{ on } \partial\Omega.$$

We have set $t_f = 1$, $\nu = 0.0055$, $K_5 = 1$, $C_4 = 0.0055$, $\varepsilon = 0.01$ and $K_3 = 100$. Moreover, we have set $m_{10} = 100$ and $\Delta t = d_2 = 1/m_{10}$.

For the solutions u, v, T, ρ obtained for the lines $j_1 = 10$, $j_1 = 50$ and $j_1 = 90$ please see figures from 1 to 12 respectively.

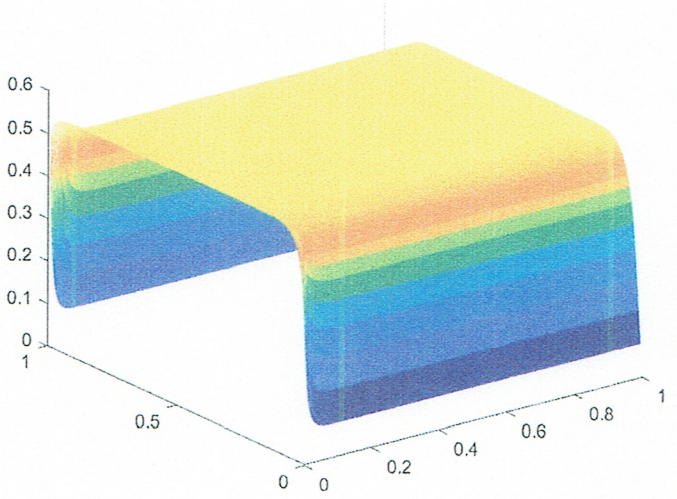

Figure 1: Velocity field $u(x,y)$ for the line $j_1 = 10$ for the case $\nu = 0.0055$.

Here we present the software in MAT-LAB entitled "LinesNavierStokesMarch2026RT1" through such numerical results have been obtained.

****

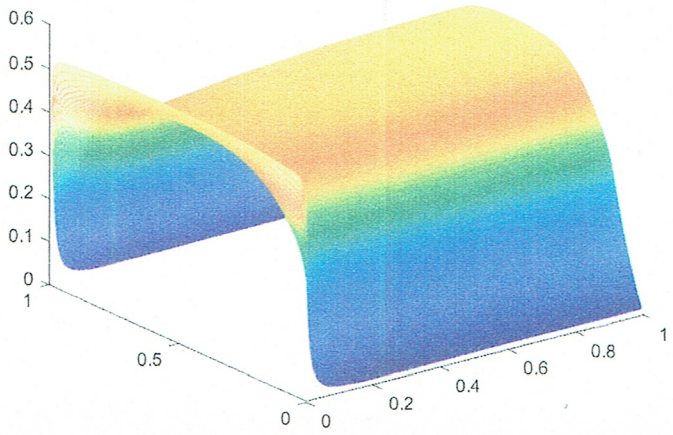

Figure 2: Velocity field $u(x, y)$ for the line $j_1 = 50$ for the case $\nu = 0.0055$.

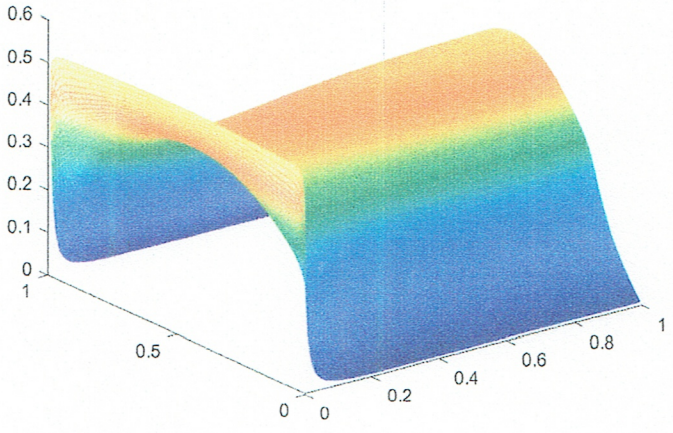

Figure 3: Velocity field $u(x, y)$ for the line $j_1 = 90$ for the case $\nu = 0.0055$.

1. clear all

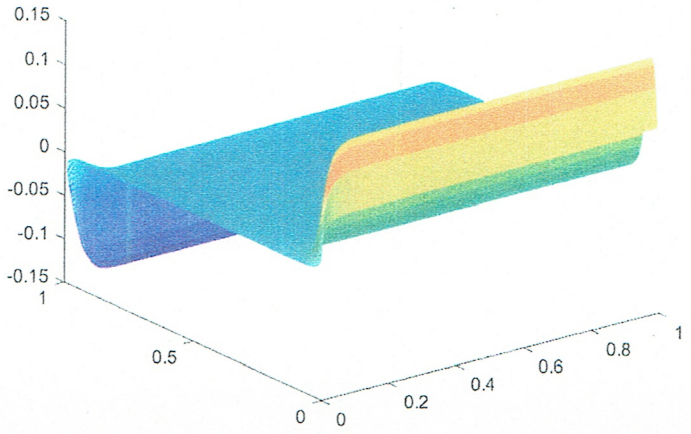

Figure 4: Velocity field $v(x, y)$ for the line $j_1 = 10$ for the case $\nu = 0.0055$.

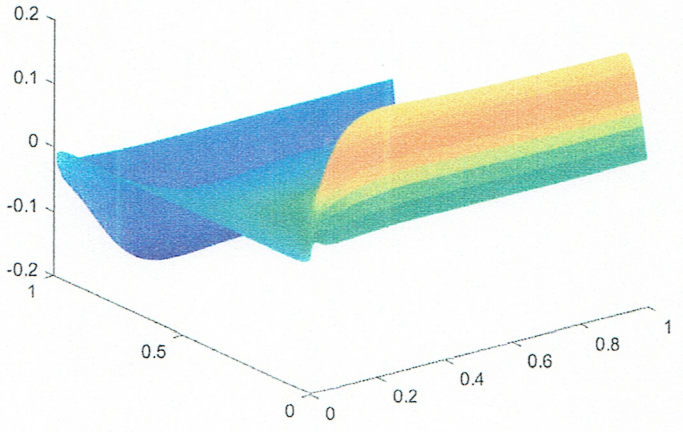

Figure 5: Velocity field $v(x, y)$ for the line $j_1 = 50$ for the case $\nu = 0.0055$.

m8=600;

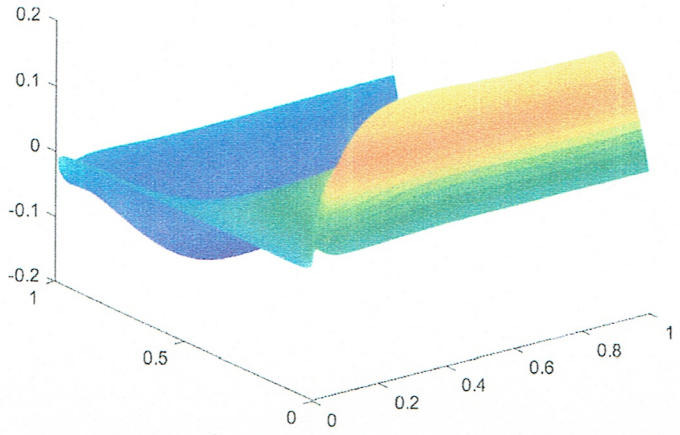

Figure 6: Velocity field $v(x, y)$ for the line $j_1 = 90$ for the case $\nu = 0.0055$.

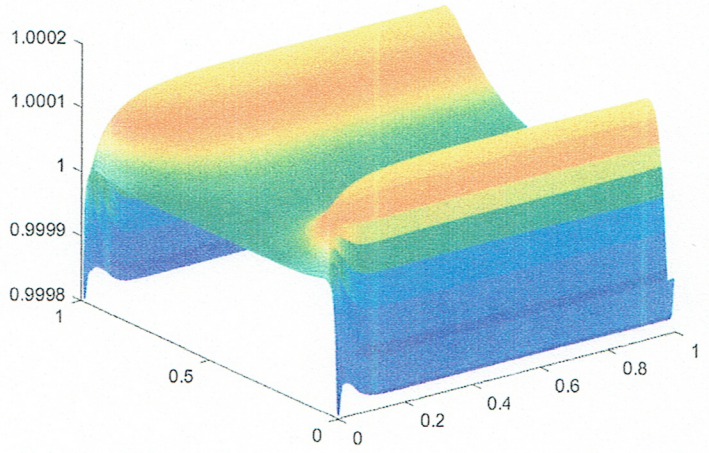

Figure 7: Temperature field $T(x, y)$ for the line $j_1 = 10$ for the case $\nu = 0.0055$.

m9=140;

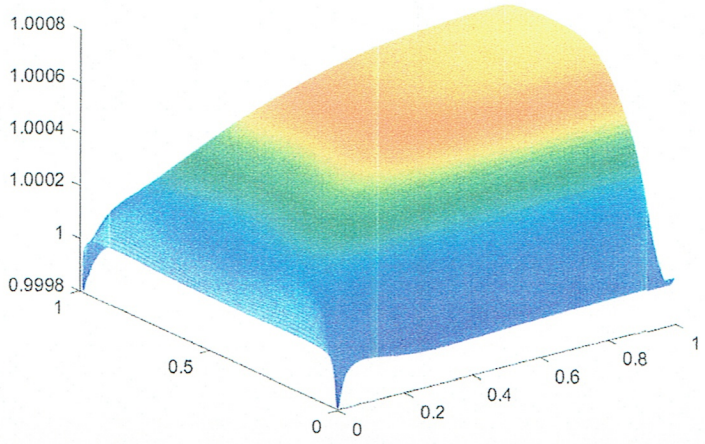

Figure 8: Temperature field $T(x, y)$ for the line $j_1 = 50$ for the case $\nu = 0.0055$.

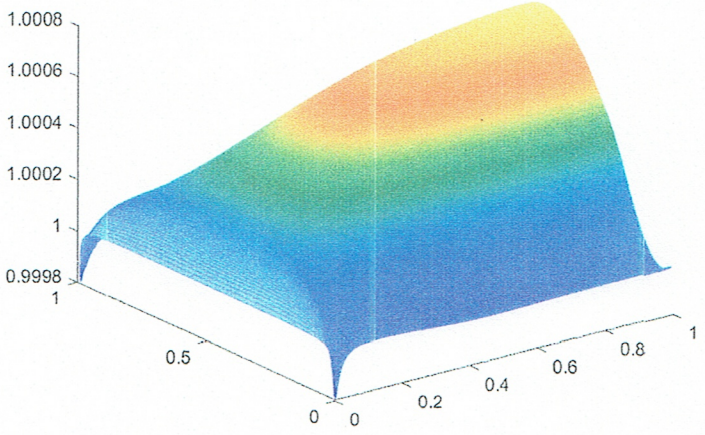

Figure 9: Temperature field $u(x, y)$ for the line $j_1 = 90$ for the case $\nu = 0.0055$.

d=1/m8;

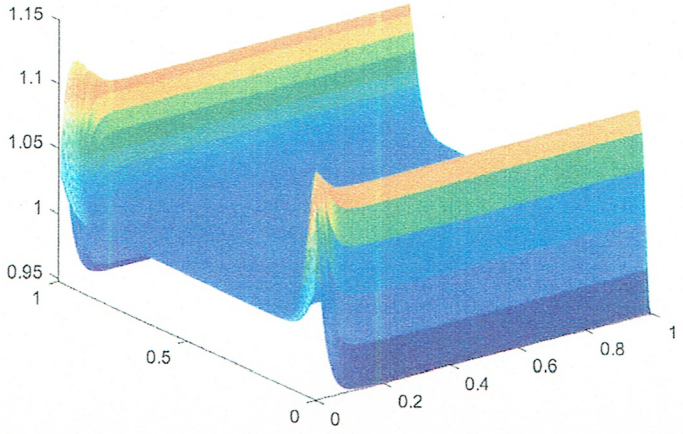

Figure 10: Density field $T(x, y)$ for the line $j_1 = 10$ for the case $\nu = 0.0055$.

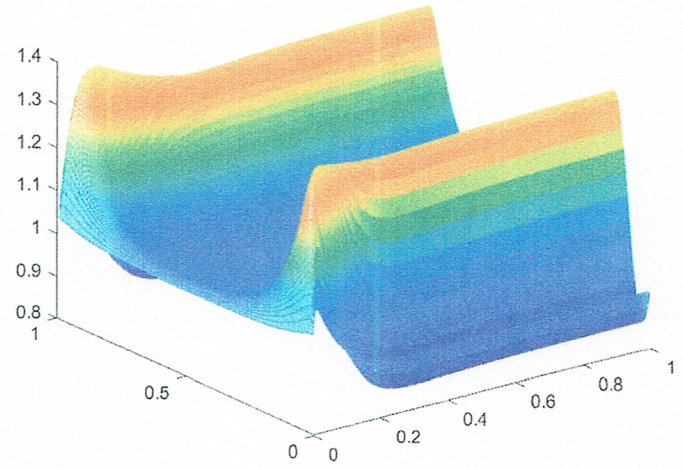

Figure 11: Density field $T(x, y)$ for the line $j_1 = 50$ for the case $\nu = 0.0055$.

d1=1/m9;

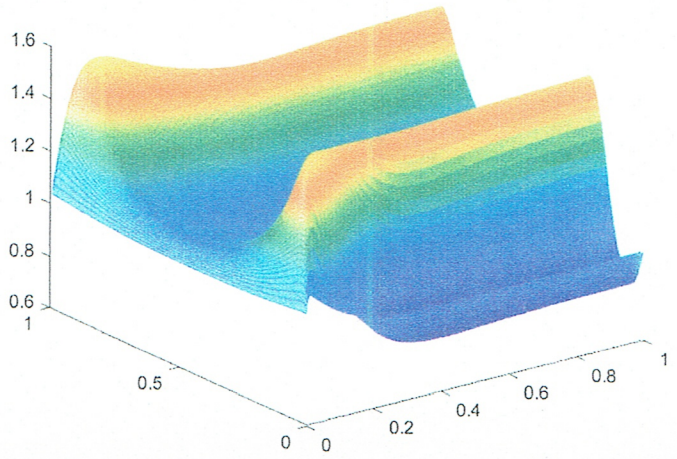

Figure 12: Density field $u(x, y)$ for the line $j_1 = 90$ for the case $\nu = 0.0055$.

```
m10=100;
d2=1/m10;
e1=0.0055;
e5=0.01;
K3=100;
C4=0.0055;
y5(:,1)=0.000000001*ones(m9-1,1);
K5=1;
Id=eye(m9-1);
m2=zeros(m9-1,m9-1);
for i=2:m9-2
m2(i,i)=-2.0;
m2(i,i+1)=1.0;
m2(i,i-1)=1.0;
end;
m2(1,1)=-2;
```

```
m2(1,2)=1;
m2(m9-1,m9-1)=-2;
m2(m9-1,m9-2)=1.0;
m1a=zeros(m9-1,m9-1);
for i=1:m9-2
m1a(i,i)=-1;
m1a(i,i+1)=1;
end;
m1a(m9-1,m9-1)=-1;
m1b=zeros(m9-1,m9-1);
for i=2:m9-1
m1b(i,i)=1;
m1b(i,i-1)=-1.0;
end;
m1b(1,1)=1;
m1=(m1a+m1b)/2;
m22=zeros(m9-1,m9-1);
for i=2:m9-2
m22(i,i)=-2.0;
m22(i,i+1)=1.0;
m22(i,i-1)=1.0;
end;
m22(1,1)=-1;
m22(1,2)=1;
m22(m9-1,m9-1)=-1;
m22(m9-1,m9-2)=1.0;
b8(:,1)=zeros(m9-1,1);
b8(1,1)=1;
b8(m9-1,1)=1;
uoo(:,1)=0.55*ones(m9-1,1);
voo(:,1)=0.0000*ones(m9-1,1);
```

```
Too(:,1)=ones(m9-1,1);
Tf(:,1)=ones(m9-1,1);
roo(:,1)=ones(m9-1,1);
for i=1:m8
uo(:,i)=0.3*ones(m9-1,1);
vo(:,i)=0.0001*ones(m9-1,1);
To(:,i)=ones(m9-1,1);
ro(:,i)=ones(m9-1,1);
u1(:,i)=0.20*ones(m9-1,1);
u3(:,1)=0.20*ones(m9-1,1);
v1(:,i)=0.01*ones(m9-1,1);
T1(:,i)=0.92*ones(m9-1,1);
r1(:,i)=0.95*ones(m9-1,1);
u(:,i,1)=uoo(:,1);
v(:,i,1)=voo(:,1);
T(:,i,1)=Too(:,1);
r(:,i,1)=roo(:,1);
end;
for j1=2:m10
b38=1;
k8=1;
while (b38 > 10^-4) && (k8 < 120)
k8
j1
k8=k8+1;
b12=1;
k=1;
while (b12 > 10^-4) && (k < 200)
k
j1
k=k+1;
```

for i=1:m8

$Po(:,i) = K3 * ro(:,i).*To(:,i) - (uo(:,i).*uo(:,i) + vo(:,i).*vo(:,i)).*ro(:,i)/2 - K3*ro(:,i);$

end;

$Poo(:,1) = K3 * roo(:,1).*Too(:,1) - (uoo(:,1).*uoo(:,1) + voo(:,1).*voo(:,1)).*roo(:,1)/2 - K3*roo(:,1);$

i=1;

$m12 = 2*Id - m2/d1^2 * d^2 + diag(ro(:,i)) * Id/d2 * d^2/e1;$

m50(:,:,i)=inv(m12);

$z1(:,i) = m50(:,:,i)*(uoo(:,1) + r(:,i,j1-1).*u(:,i,j1-1)/d2*d^2/e1 - uo(:,i).*(ro(:,i).*uo(:,i) - roo(:,1).*uoo(:,1))/d*d^2/e1);$

$z1(:,i) = z1(:,i) + m50(:,:,i) * (-vo(:,i).*(m1*(ro(:,i).*uo(:,i)))/d1 * d^2/e1 - (Po(:,i) - Poo(:,1))/d*d^2/e1);$

for i=2:m8-1

$m12 = 2*Id - m2/d1^2 * d^2 + diag(ro(:,i)) * Id/d2 * d^2/e1 - m50(:,:,i-1);$

m50(:,:,i)=inv(m12);

$z1(:,i) = m50(:,:,i)*(r(:,i,j1-1).*u(:,i,j1-1)/d2*d^2/e1 - uo(:,i).*(ro(:,i).*uo(:,i) - ro(:,i-1).*uo(:,i-1))/d*d^2/e1);$

$z1(:,i) = z1(:,i) + m50(:,:,i) * (z1(:,i-1) - vo(:,i).*(m1*(ro(:,i).*uo(:,i)))/d1 * d^2/e1 - (Po(:,i) - Po(:,i-1))/d*d^2/e1);$

end;

u(:,m8,j1)=inv(Id-m50(:,:,m8-1))*z1(:,m8-1);

for i=1:m8-1

u(:,m8-i,j1)=m50(:,:,m8-i)*u(:,m8-i+1,j1)+z1(:,m8-i);

end;

uo=u(:,:,j1);

b12=max(max(abs(uo-u1)));

u1=uo;

end;

b14=1.0;

k1=1;

while $(b14 > 10^{-4})$ && $(k1 < 200)$

k1

j1

k1=k1+1;

i=1;

$m14 = 2 * Id - m2/d1^2 * d^2 + diag(ro(:,i)) * Id/d2 * d^2/e1;$

m60(:,:,i)=inv(m14);

$z2(:,i) = m60(:,:,i) * (voo(:,1) + r(:,i,j1-1). * v(:,i,j1-1)/d2 * d^2/e1 - uo(:,i). * (ro(:,i). * vo(:,i) - roo(:,1). * voo(:,1))/d * d^2/e1);$

$z2(:,i) = z2(:,i) + m60(:,:,i) * (-vo(:,i). * (m1 * (ro(:,i). * vo(:,i)))/d1 * d^2/e1 - m1 * Po(:,i)/d1 * d^2/e1);$

for i=2:m8-1

$m14 = 2 * Id - m2/d1^2 * d^2 + diag(ro(:,i)) * Id/d2 * d^2/e1 - m60(:,:,i-1);$

m60(:,:,i)=inv(m14);

$z2(:,i) = m60(:,:,i) * (r(:,i,j1-1). * v(:,i,j1-1)/d2 * d^2/e1 - uo(:,i). * (ro(:,i). * vo(:,i) - ro(:,i-1). * vo(:,i-1))/d * d^2/e1);$

$z2(:,i) = z2(:,i) + m60(:,:,i) * (z2(:,i-1) - vo(:,i). * (m1 * (ro(:,i). * vo(:,i)))/d1 * d^2/e1 - m1 * Po(:,i)/d1 * d^2/e1);$

end;

v(:,m8,j1)=inv(Id-m60(:,:,m8-1))*z2(:,m8-1);

for i=1:m8-1

v(:,m8-i,j1)=m60(:,:,m8-i)*v(:,m8-i+1,j1)+z2(:,m8-i);

end;

vo=v(:,:,j1);

b14=max(max(abs(vo-v1)));

v1=vo;

end;

i=1;

$W8(:,i) = 2 * C4 * (uo(:,i) - uoo(:,1))/d. * (uo(:,i) - uoo(:,1))/d + 2 * C4 * m1 * vo(:,i)/d1. * (m1 * vo(:,i)/d1);$

$W8(:,i) = W8(:,i) + C4 * (m1 * uo(:,i)/d1). * (m1 * uo(:,i)/d1) + C4 * (vo(:,i) - voo(:,1))/d. * (vo(:,i) - voo(:,1))/d;$

$W8(:,i) = W8(:,i)-2/3*C4*((uo(:,i)-uoo(:,1))/d+m1*vo(:,i)/d1).*((uo(:,i)-uoo(:,1))/d+m1*vo(:,i)/d1);$

for i=2:m8-1

$W8(:,i) = 2*C4*(uo(:,i)-uo(:,i-1))/d.*(uo(:,i)-uo(:,i-1))/d+2*C4*m1*vo(:,i)/d1.*(m1*vo(:,i)/d1);$

$W8(:,i) = W8(:,i)+C4*(m1*uo(:,i)/d1).*(m1*uo(:,i)/d1)+C4*(vo(:,i)-vo(:,i-1))/d.*(vo(:,i)-vo(:,i-1))/d;$

$W8(:,i) = W8(:,i)-2/3*C4*((uo(:,i)-uo(:,i-1))/d+m1*vo(:,i)/d1).*((uo(:,i)-uo(:,i-1))/d+m1*vo(:,i)/d1);$

end;

b16=1.0;

k2=1;

while $(b16 > 10^{-4})$ && $(k2 < 200)$

k2

j1

k2=k2+1;

i=1;

$m16 = 2*Id-m2/d1^2*d^2+diag(ro(:,i))*Id/d2*d^2/K5;$

m70(:,:,i)=inv(m16);

$z3(:,i) = m70(:,:,i)*(b8(:,1)*d^2/d1^2+Too(:,1)+ro(:,i).*T(:,i,j1-1)/d2*d^2/K5+Po(:,i).*((uo(:,i)-uoo(:,1))/d+m1*vo(:,i)/d1)*d^2/K5-W8(:,i)*d^2/K5);$

for i=2:m8-1

$m16 = 2*Id-m2/d1^2*d^2+diag(ro(:,i))*Id/d2*d^2/K5-m70(:,:,i-1);$

m70(:,:,i)=inv(m16);

$z3(:,i) = m70(:,:,i)*(b8(:,1)*d^2/d1^2+z3(:,i-1)+ro(:,i).*T(:,i,j1-1)/d2*d^2/K5+Po(:,i).*((uo(:,i)-uo(:,i-1))/d+m1*vo(:,i)/d1)*d^2/K5-W8(:,i)*d^2/K5);$

end;

T(:,m8,j1)=inv(Id-m70(:,:,m8-1))*z3(:,m8-1);

for i=1:m8-1

T(:,m8-i,j1)=m70(:,:,m8-i)*T(:,m8-i+1,j1)+z3(:,m8-i);

end;

To=T(:,:,j1);

b16=max(max(abs(To-T1)));

T1=To;

end

b30=1;

k5=1;

while $(b30 > 10^{-4})$ && $(k5 < 200)$

k5

j1

k5=k5+1;

i=1;

$m30 = 2 * Id - m2/d1^2 * d^2 + Id/d2 * d^2/e5;$

m80(:,:,i)=inv(m30);

$z5(:,i) = m80(:,:,i) * (b8(:,1) * d^2/d1^2 + roo(:,1) + r(:,i,j1-1)/d2 * d^2/e5 - (ro(:,i) - roo(:,1))/d. * uo(:,i) * d^2/e5 - m1 * ro(:,i)/d1. * vo(:,i) * d^2/e5);$

$z5(:,i) = z5(:,i) + m80(:,:,i) * (-ro(:,i). * ((uo(:,i) - uoo(:,1))/d + m1 * vo(:,i)/d1) * d^2/e5);$

for i=2:m8-1

$m30 = 2 * Id - m2/d1^2 * d^2 + Id/d2 * d^2/e5 - m80(:,:,i-1);$

m80(:,:,i)=inv(m30);

$z5(:,i) = m80(:,:,i) * (z5(:,i-1) + b8(:,1) * d^2/d1^2 + r(:,i,j1-1)/d2 * d^2/e5 - (ro(:,i) - ro(:,i-1))/d. * uo(:,i) * d^2/e5 - m1 * (ro(:,i)/d1). * vo(:,i) * d^2/e5);$

$z5(:,i) = z5(:,i) + m80(:,:,i) * (-ro(:,i). * ((uo(:,i) - uo(:,i-1))/d + m1 * vo(:,i)/d1) * d^2/e5);$

end;

r(:,m8,j1)=inv(Id-m80(:,:,m8-1))*z5(:,m8-1);

for i=1:m8-1

r(:,m8-i,j1)=m80(:,:,m8-i)*r(:,m8-i+1,j1)+z5(:,m8-i);

end;

ro=r(:,:,j1);

b30=max(max(abs(ro-r1)));

```
r1=ro;
end;
b38=max(max(abs(u3-u1)));
u3=u1;
end;
end;
for i=1:m8
x1(i,1)=i*d;
end;
for j=1:m9-1
y1(j,1)=j*d1;
end;
mesh(x1,y1,u(:,:,90))
```

****

5 Conclusion

In this article, we have developed a variational formulation for modeling an atomic hydrogen gas motion in an Euler-Bernoullian context.

In summary, we have obtained a functional such that the variation comprises a set of partial differential equations intended to mathematically model such a fluid motion.

We have obtained the establishment of a system of partial differential equations suitable for compressible fluid mechanics, including a Bernoulli-perfect-gas type equation as previously indicated.

Finally, in the last section we have developed a closely related numerical example.

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